Reply to "Comment on 'Self-organized Criticality and Absorbing States: Lessons from the Ising Model'"

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In [Braz. J. Phys. **30**, 27 (2000)] Dickman *et al.* suggested that self-organized criticality can be produced by coupling the activity of an absorbing state model to a dissipation mechanism and adding an external drive. We analyzed the proposed mechanism in [Phys. Rev. E **73**, 025106(R) (2006)] and found that if this mechanism is at work, the finite-size scaling found in self-organized criticality will depend on the details of the implementation of dissipation and driving. In the preceding comment [Phys. Rev. E **XX**, XXXX (2008)], Alava *et al.* show that one avalanche exponent in the AS approach becomes independent of dissipation and driving. In our reply we clarify their findings and put them in the context of the original article.

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In [1] we discussed the implications of the absorbing state (AS) mechanism [2] if it were to underlie self-organized criticality (SOC). One of the key ingredients of the AS-mechanism is that dissipation and driving are made to vanish in the thermodynamic limit. We showed that criticality would be reached in the thermodynamic limit for almost any choice of the scaling with system size of dissipation and driving (with the effective temperature vanishing, contrary to what is stated in [3]). While this choice is thus not important to answer the question *if* the critical point will be reached, it is important when addressing the question *how* it will be reached. In particular, we showed that the critical point would be reached in the limit of slow drive $\omega - \kappa > \beta/\nu$ ("too fast" in [1]), but that the observed finite-size scaling exponents would depend on $\omega - \kappa$, and the relative correlation length ξ/L would vanish asymptotically.

The present discussion can be phrased in terms of two statements. 1) SOC is universal. 2) The AS-mechanism is solely responsible for SOC. In [1] we showed that these are mutually exclusive. If the AS-mechanism is the reason for SOC, then SOC cannot be universal. This would weaken SOC considerably since studying models as simple as sandpiles is only sensible if these systems show universal behavior. The other possibility is that SOC is universal, which would weaken the AS-mechanism, because it does not predict universality. Our analysis was restricted to the finite-size scaling of AS observables, such as the order parameter, the correlation length, and the susceptibility, whereas the preceding comment [3] by Alava *et al.* refers to avalanche characteristics. In the following we assume that what we found out about the universality of AS observables also applies to avalanches. We emphasize that we do not know whether this is true; this assumption is made in order to be able to reply to the comment, which makes the same assumption.

In the following we distinguish between the AS approach intended to explain SOC, and SOC itself. Alava *et al.* do not make this distinction explicit. In the AS approach driving and dissipation rates are tuned (bulk dissipation as $L^{-\kappa}$ and driving as $L^{-\omega}$), whereas in SOC (boundary dissipation, driving on a separate time scale) they are set implicitly by the dynamics of the system. If the AS approach applies to SOC, then SOC behaviour is obtained within the AS approach by taking the thermodynamic limit. Both, our original article [1] as well as the preceding comment [3], are concerned solely with the characterisation of the AS approach.

It is important to stress that in [1] we did *not* claim that any exponents, neither those describing avalanches nor those characterising the activity, in standard SOC models are non-universal. This deserves clarification, because the opposite is stated in the abstract of [3]. As discussed in [1], there are instances of SOC exponents being identical across a wide range of models [4, 5, 6], while others are not [5, 6].

We do claim (with the proviso stated above), however, that avalanche size exponents would be non-universal if the AS mechanism [2] was applicable to SOC [1]. Alava *et al.* challenge this claim by showing that *within the AS approach* τ_s is independent of the scaling of driving and dissipation in the slow driving limit. This limit corresponds to the separation of time scales in SOC.

In [1] we explicitly mention avalanche exponents only once: "In the AS approach also the avalanche size exponents show a clear, immediate dependence on the choice of the two exponents κ and ω." It is correct that the avalanche size exponents τ_s and D_L (see below) depend on κ and ω, but as Alava *et al.* show, for ω large enough τ_s becomes independent of κ and ω. In this case, as we show below, D_L nonetheless depends on κ. The non-universality of D_L within the AS approach implies that it does not explain universal SOC. This is the same conclusion we reached by studying AS observables (order parameter, correlation length, susceptibility).

We agree that the derivation in [1] necessarily breaks down in the large ω limit. We explicitly assumed finite bounds for both, κ and ω, and we did not discuss the case of ω being greater than the dynamical exponent, which is the regime studied by Alava *et al.*. We agree that this regime is the most important one for SOC.

Alava *et al.* have chosen an observable that is independent of the external drive for sufficiently large ω, but its finite-size scaling turns out to depend on the scaling of the dissipation. This can be seen by a finite-size scaling analysis of the characteristic avalanche size s_c. According to Ref. [3], Eq. (3), $s_c \propto \xi_s^D$, where ξ_s is some "cut-off scale". Usually, the exponent D is reserved for the finite size scaling of s_c, also known as the avalanche dimension [5], which we call D_L in the following, so that $s_c \propto L^{D_L}$. Combining Eq. (4) and Eq. (3) of [3], they find $D_L = \kappa/(2-\tau_s)$ with τ_s being independent of external drive and dissipation. This is a very surprising result, because regardless of whether or not the AS-mechanism applies, the avalanche dimension D_L is deeply rooted in the model and in general directly related to the field theory of the corresponding depinning transition [5, 7, 8, 9]. There are several models [4, 5, 6] which display a universal avalanche dimension D_L and an avalanche size exponent $\tau_s = 2 - \gamma_1/D_L$ which depends on the details of the driving of the model [5], with the first moment scaling like $\langle s \rangle \propto L^{\gamma_1}$. For these conservative models, it is straight forward to devise a method to produce any exponent τ_s in the interval $[2-2/D_L, 2-1/D_L]$ by effectively tuning γ_1. This can be achieved by changing the driving mechanism, [10] as the driving mechanism leaves D_L unchanged.

We would have expected that changing γ_1 by introducing dissipation would have the same effect, *i.e.* varying τ_s and constant D_L. However, Alava *et al.* find that D_L depends on κ, while τ_s remains unchanged, a very interesting numerical finding we do not dispute. It has thus been

established that, under the appropriate conditions, both τ_s and D_L can be tuned. Importantly, D_L can be tuned in the SOC-regime of the AS approach (large ω) by changing κ.

Inasmuch as avalanche exponents pertain to the discussion the universality of τ_s within the AS approach supports the case for universal SOC being generated by the AS mechanism. But, the implicit finding by Alava *et al.* that D_L depends on κ confirms that, apparently, the AS approach does not produce universal SOC.

In order to address the question whether the AS-mechanism is operating in SOC models (universal or not), one needs to probe its presence either directly or test its implications. In [1] we have shown that the AS-mechanism would (almost always) imply a vanishing relative correlation length ξ/L and a finite-size scaling of the AS order parameter, characterized by exponents β/μ and γ/μ, that would depend on the scaling of dissipation and drive, parameterized by κ and ω. We stated explicitly how β/μ and γ/μ depend on κ and ω, while further analysis is necessary for the dependence of the avalanche exponents on κ and ω. At the present stage, a more promising route than studying avalanches therefore seems to be the study of AS observables in SOC systems.

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[1] G. Pruessner and O. Peters, Phys. Rev. E **73**, 025106(R) (2006), cond-mat/0411709.

[2] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000), cond-mat/9910454v2.

[3] M. Alava, L. Laurson, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. (2008), preceding comment.

[4] G.-J. Pan, D.-M. Zhang, H.-Z. Sun, and Y.-P. Yin, Commun. Theor. Phys. **44**, 483 (2005).

[5] K. Christensen, N. R. Moloney, O. Peters, and G. Pruessner, Phys. Rev. E **70**, 067101 (2004), cond-mat/0405454.

[6] A. Malthe-Sørenssen, Phys. Rev. E **59**, 4169 (1999).

[7] M. Paczuski and S. Boettcher, Phys. Rev. Lett. **77**, 111 (1996).

[8] G. Pruessner, Phys. Rev. E **67**, 030301(R) (2003), cond-mat/0209531.

[9] M. Alava, in *Advances in Condensed Matter and Statistical Physics*, edited by E. Korutcheva and R. Cuerno (Nova Science Publishers, New York, 2004), ISBN 1-59033-899-5.

[10] As long as the model is conservative, individual particles effectively perform a random walk and the first moment of the avalanche size is a function of the spatial distribution of the driving. For example, if a one-dimensional model is driven at site x_0 with two open boundaries, then the first moment is $(L+1-x_0)x_0$ where L is the system size. By changing x_0 as a power law of L, the exponent γ_1 changes accordingly. For example, $x_0 = \sqrt{L}$, produces $\gamma_1 = 3/2$.